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A Numerical Study of Natural Convection in the Horizontal Bridgman Configuration under an External Magnetic Field

Nizar Ben Salah [a], Inès Baaziz [a], and Daniel Rousse [b,*]
[a]*LMMP, Ecole Supérieure des Sciences et Techniques De Tunis (ESSTT), 1008 Tunis, Tunisia*
[b] *Department of applied sciences, Université du Québec à Chicoutimi, Chicoutimi, G7H 2B1, Canada*

Abstract

Plasma confinement, liquid-metal cooling of nuclear reactors, and electromagnetic casting are among the several engineering applications of magneto-hydrodynamics. This study presents a numerical method that allows predicting the electromagnetic braking of free-convective flows in cavities used in the crystal growth horizontal Bridgman configuration. The cavities are filled with dilute electrically conducting alloys, with a low Prandtl number, and subjected to a horizontal temperature gradient. The flow is assumed to be steady and laminar and subject to an external vertical, transversal, and uniform magnetic field. The cavities are assumed to be infinite horizontal arbitrarily-shaped rods.

A numerical code based on a finite element method (FEM) has been formulated and implemented to study the effects of the variation of the Hartmann number, on the velocity profiles, and on the induced magnetic field for a given Grashoff number, and for several cross section shapes (namely circle, square, and lozenge). The proposed FEM and its implementation are validated through a thorough comparison with other studies.

The numerical results obtained for the variation of the velocity and the magnetic fields in terms of the Hartmann number, show a considerable decrease in convective intensity as the Hartmann number increases, and reveal the progressive building of the well known Hartmann and parallel layers.

Keywords: Free convection, magnetic field, magneto-hydrodynamics, arbitrarily-shaped cross sections, solidification.

1. Introduction

In the field of the elaboration of metal alloys, one of the main stakes lives in the mastering of the metallurgical structure and of the potential defects which appear during the solidification phase. A controlled metallurgical structure allows guaranteeing certain properties of the solidified material. Indeed, in the majority of the methods of steered solidification used for the elaboration of high quality single metal alloys crystals, such as the Czochralski or Bridgman configurations, the crystal develops slowly within a fluid fed in molds having variable geometries and which is subjected to a unidirectional temperature gradient from the hot zone towards the cold one. This gradient gives birth to unavoidable natural convection movements, in the liquid phase of the melt, which acts on the crystalline growth by ordering the transport of heat and doping impurities phenomena at the growth interface. Actually, the appearance of convection during the crystalline growth can lead to non-homogeneities which induce striations and defects and thus affect the quality of crystals. Moreover, the oscillations of temperature due to the instabilities of the flow lead to a non-uniform cooling at the front of the solidification. Then, naturally, understanding the natural convection movements and being able to control them become preponderant.

[*] Corresponding author. Tel.: +1 (418) 657-4428 Fax: +1 (418) 657-2132; E-mail: Daniel.Rousse@uquebec.ca

In the context of crystal growth, natural convection was the object of many studies during the last three decades. One can consult the pioneering works of Carruthers [1], Hurle [2], Kobayashi [3], and Langlois [4]. The control of the intensity of the convection can be realized by exposing the convector flow to a magnetic field of constant strength and direction. This has been put into evidence in many works with the most significant ones are proposed by Garandet *et al.* [5], Alchaar *et al.* [6], Ben Hadid and Henry [7], Davoust *et al*. [8] and Bùhler [9]. However, one can notice that very few studies were interested in the effect of the cavity cross section shape on the velocity damping and on the induced magnetic field. As far as we are concerned, the only available works involving this particularity are those of Garandet et al. [5] and Moreau *et al*. [10] which consider different cross section shapes. In these works, the authors present an asymptotic analytical solution for the flow, in the core region, developed in the high Hartmann numbers limit and a numerical solution using a commercial code FLUX-EXPERT [11].

The purpose of the present work is to develop a numerical finite element method, in order to study the influence of the application of a vertical magnetic field on the phenomenon of the natural convection during the solidification of a liquid metal in the artificial crystal growth horizontal Bridgman configuration. Numerical results cover several symmetric cross section shapes (circle, square, and lozenge) and are obtained for perfectly electrically and thermally insulating walls.

The rest of the paper is organized in five sections:

- In the second section, the mathematical modeling of the studied problem is presented. Essentially, the phenomenon can be described by partial differential equations coupled with an integral flow conservation equation. Under the assumptions stated for the problem, these equations can be summarized as two Poisson equations coupled through their right hand side members.
- The third section briefly presents the dimensionless model which stresses attention on the relevant parameters that govern the phenomenon.
- In the next section, the weak integral form of the problem and the usual associated discretization process are discussed.
- In the fifth section, some of the most important results are presented. These results mainly show the development of the electromagnetic braking of the flow and the building of the Hartmann and parallel layers, as the Hartmann number increases.
- Discussions of results and main conclusions are drawn in the sixth and last section.

2. Mathematical description

The studied configuration is an infinite horizontal rod for which the cross-section S is defined by the equations $z_1(y)$ and $z_2(y)$ of its upper and lower half contours. The geometry is supposed to be filled with an electrically conducting liquid and submitted to an externally applied vertical uniform magnetic field $\vec{B}_0$ (see Fig. 1).

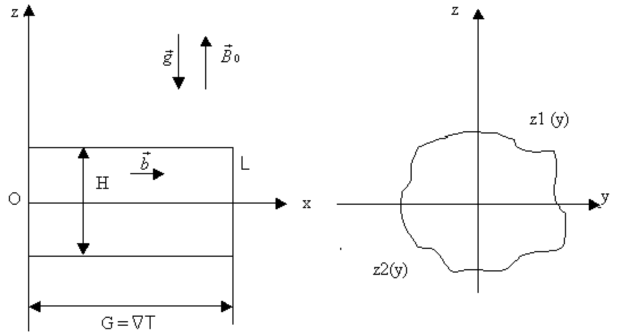


Figure 1: Studied configuration in present work

In this work, we are only concerned with the fully established regime of zero flow rate driven by buoyancy. The velocity $\vec{u}$ and the induced magnetic field $\vec{b}$ which is superposed to the applied magnetic field $\vec{B}_0$ are then collinear and directed along the horizontal axis Ox. Under the Boussinesq approximation and in the limit of low Prandtl number, which is completely justified for liquid metals, the velocity and the induced magnetic field satisfy the following mathematical description involving three linear equations of motion and induction.

$$\vec{\nabla}.\vec{u} = 0 \qquad (1)$$

$$\rho_0\, \nu\, \nabla^2\vec{u} = \vec{\nabla}p - \vec{J}\times\vec{B}_0 + \rho_0\beta G x \vec{g} \qquad (2)$$

$$\nabla^2\vec{b} = -\mu\,\sigma\,(\vec{B}_0\cdot\vec{\nabla})\vec{u} \qquad (3)$$

Where p, ν, μ, σ, ρ_0, β and $\vec{J}$ denote respectively the total pressure, the kinematic viscosity, the magnetic permeability, the electrical conductivity, the density, the thermal dilatation coefficient, and the electric current density. It is noticeable that this set of equations does not require any assumption on the value of the magnetic Reynolds number, since $(\vec{b}.\vec{\nabla})$ is identically zero and that $\vec{J}\times\vec{b}$ is a pure gradient incorporated into the definition of the pressure such that:

$$p = p_{stat} + \rho_0 g z + \frac{b^2}{2\mu} \qquad (4)$$

Integrating the projection of equation (2) in the z direction yields

$$p = -\rho_0\beta G x[z-\bar{z}]g \qquad (5)$$

Where the constant $\bar{z}$ has to be such that total flow rate in the cross section S is zero:

$$\iint_S u\, dydz = 0 \qquad (6)$$

The equations of the problem can then be summarized as two Poisson equations coupled through their right hand side terms, together with the equation (6), which implicitly determines the constant $\bar{z}$:

$$\Delta u = -\frac{\beta G}{\nu}(z-\bar{z}) - \frac{B_0}{\rho_0\mu\nu}\frac{\partial b}{\partial z} \qquad (7)$$

$$\Delta b = -\mu\sigma B_0\frac{\partial u}{\partial z} \qquad (8)$$

The boundary conditions are classical. On the contour of section S they imply the no-slip condition $(u=0)$ and the continuity of $\boldsymbol{b}$. In this work, we limit ourselves to the study of perfectly insulating walls, then, by virtue of Ampere's law $\boldsymbol{b}$ =0.

3. Dimensionless model

It is convenient to use non-dimensional variables based on a typical length H of the section S , on B_0 and on the physical properties of the fluid. For our problem, we prefer to introduce the following variables:

$$Y = y/H,\ Z = z/H,\ U = \frac{u\,H}{\nu},\ B = \frac{b}{B_0\mu\sigma\nu}$$

Then, the non dimensional model for the problem involves the following set of equations:

$$\Delta U - Gr(Z-\bar{Z}) + Ha^2\frac{\partial B}{\partial z} = 0 \qquad (9)$$

$$\Delta B + \frac{\partial U}{\partial z} = 0 \qquad (10)$$

$$\iint_S U\, dY dZ = 0 \tag{11}$$

Where Ha and Gr are respectively the dimensionless Hartmann and Grashoff numbers defined such that:

$$Ha = \sqrt{\frac{\sigma}{\rho\nu}} B_0 H \quad , \qquad Gr = \frac{\rho_0 g \beta G H^4}{\nu^2}$$

In general, such an integral-differential set of equations might require long iterative procedure to determine the constant $\bar{Z}$. In this particular case, however, the difficulty is easily overcome, because the flow rate is a linear function of $\bar{Z}$. For symmetric sections $\bar{Z}$ is simply zero. So, the equations of interest can finally be written as two coupled Poisson equations:

$$\Delta U - Gr\, Z + Ha^2 \frac{\partial B}{\partial Z} = 0 \tag{12}$$

$$\Delta B + \frac{\partial U}{\partial Z} = 0 \tag{13}$$

4. Variational formulation

The strong Galerkin integral formulations of the problem are obtained by multiplying equations (12) and (13) by two test functions v_1 and v_2 belonging to $H_0^1(\Omega)$ and by integrating over the domain. The weak Galerkin formulations are then obtained after integration by parts of the second order terms and can be written as:

$$\int_\Omega \left[Ha^2 \frac{\partial B}{\partial Z} v_1 - \left(\frac{\partial U}{\partial Y}\frac{\partial v_1}{\partial Y} + \frac{\partial U}{\partial Z}\frac{\partial v_1}{\partial Z}\right)\right] dY dZ = \int_\Omega Gr\, Z\, v_1\, dY dZ \tag{14}$$

$$\int_\Omega \left[-\left(\frac{\partial B}{\partial Y}\frac{\partial v_2}{\partial Y} + \frac{\partial B}{\partial Z}\frac{\partial v_2}{\partial Z}\right) + \frac{\partial U}{\partial Z} v_2\right] dY dZ = 0 \tag{15}$$

As usual, the associated discrete problem can be stated as:

$$\sum_{\Omega e} \int_{\Omega e} \left[Ha^2 \frac{\partial B_h}{\partial Z_h} {v_1}_h - \left(\frac{\partial U_h}{\partial Y_h}\frac{\partial v_{1h}}{\partial Y_h} + \frac{\partial U_h}{\partial Z_h}\frac{\partial v_{1h}}{\partial Z_h}\right)\right] dY dZ = \int_{\Omega e} Gr\, Z_h\, {v_1}_h\, dY dZ \tag{16}$$

$$\sum_{\Omega e} \int_{\Omega e} \left[-\left(\frac{\partial B_h}{\partial Y_h}\frac{\partial v_{2h}}{\partial Y_h} + \frac{\partial B_h}{\partial Z_h}\frac{\partial v_{2h}}{\partial Z_h}\right) + \frac{\partial U_h}{\partial Z_h} {v_2}_h\right] dY dZ = 0 \tag{17}$$

The integral form (16)-(17) has been interpolated on linear triangular elements. The unstructured mesh is concentrated near the walls where the Hartmann and the parallel layers are supposed to develop and since their thickness (Ha^{-1} for Hartmann layers and $Ha^{-1/2}$ for parallel layers) diminishes with increasing Hartmann numbers, the grid densities have been chosen according to the value of the Hartmann number Ha. Selected grids involving more and more nodes where then used to ensure that, for the final one retained here, the numerical solution is insensitive to further grid refinements.

5. Numerical predictions

The first numerical test was intended to validate the formulation, the discretization, and the implementation of the proposed finite element method. The numerical solution for the velocity filed was obtained for the case of a rod of circular cross section with *Ha* = 10^{-12} (that is no magnetic field) and *Gr*=10^6 (to compare with a benchmark solution) (see Fig. 2). This numerical solution was then compared with the acknowledged analytical solution presented by Bejan *et al* [12] for fully developed natural convection flows in long circular pipes subject to temperature gradients with the same Grashoff number. As expected, the two solutions are in perfect agreement thus enabling the method for numerical analysis.

Then, the interest was put on the variation of the cross-section shape. Three different shapes in the Y-Z planes were considered: square, circle and lozenge (see Fig. 3) and for the rest of the results presented herein, the Grashoff number is chosen to be: *Gr*=10^5.

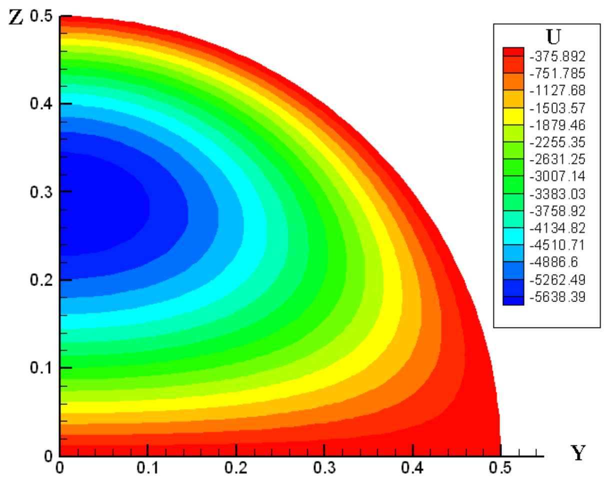


Fig. 2: Velocity contours for *Ha* = 10^{-12} and *Gr*=10^6.

Since equations (12) and (13) are linear and that the investigated sections are symmetric with respect to axes Y and Z, the solutions U and B are also symmetric with respect to Z and anti-symmetric with respect to Y. As a result, the solution can be obtained only for a quarter of the domain (see Fig. 3). Fig. (4), (5), and (6) show the contours of the non dimensional velocity field respectively in circular, square and lozenge cross sections.

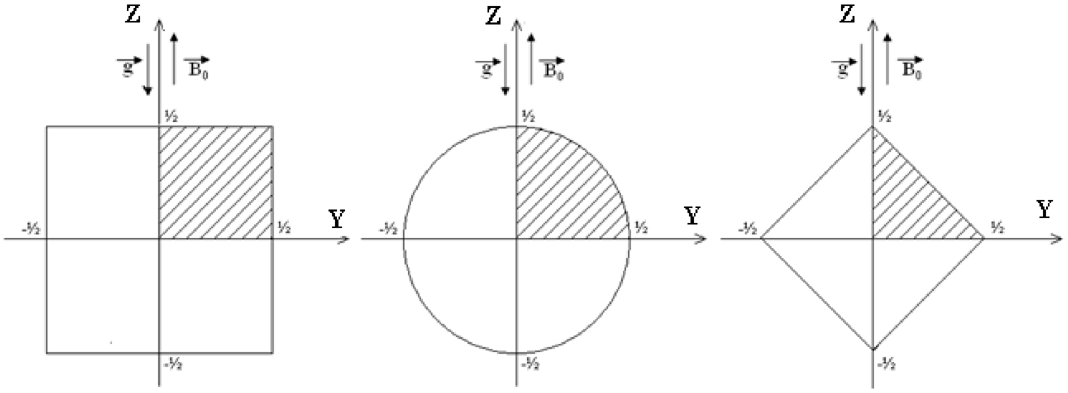
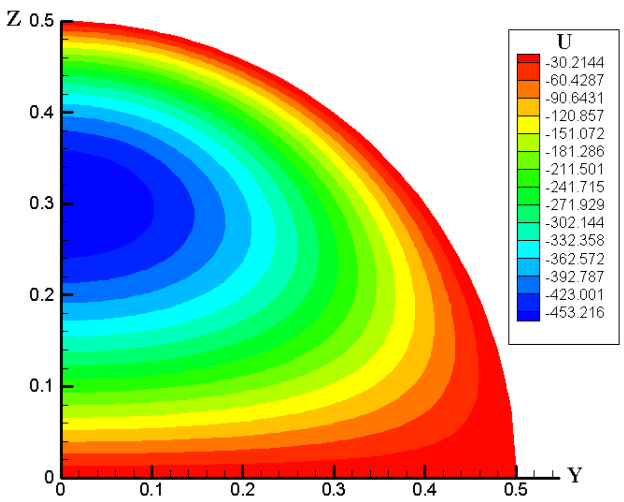
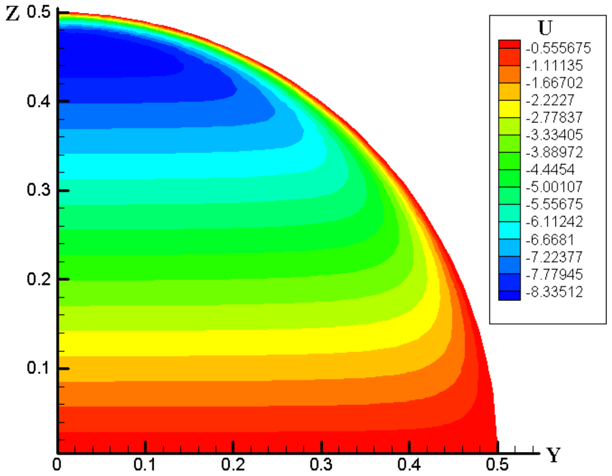


Fig. 3: Cross section shapes of the infinite rods investigated in this study and relevant notation

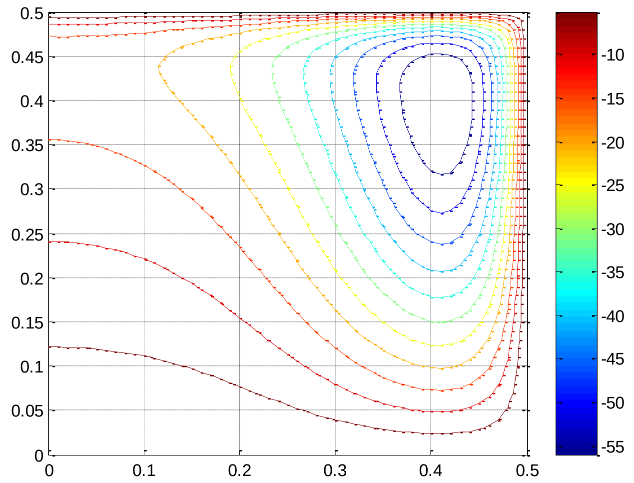


(a)

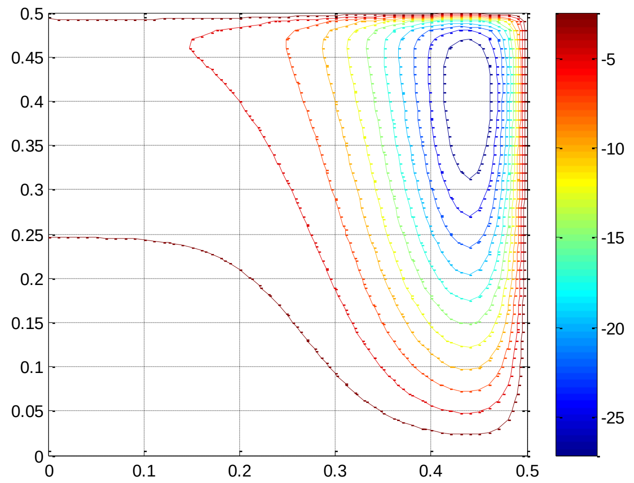


(b)

Fig. 4: Velocity field for the circular rod: a) *Ha* =5 ; and (b) *Ha* = 100



(a)



(b)

Figure 5: Velocity field for the square rod: a) Ha =50 ; and (b) Ha = 100

The electromagnetic braking of the Lorentz forces is obvious on all figures. Both for the circular, square, and lozenge-shaped sections, for which results are presented in Fig. 3, 4, and 5, respectively, as the Hartmann number increases, which is a direct result of the increase of the magnitude of the magnetic field, one can observe the drastic reduction of the velocity magnitude. Moreover, the development of the Hartmann layers for the three cross sections is shown in the figures. These layers which develop in the normal direction of the magnetic field have a thickness of the order of Ha^{-1}. For the square section, an over velocity is present, near the boundaries that are parallel to the magnetic field direction. These over velocities are the signature of the presence of two parallel layers having a thickness of $Ha^{-1/2}$. Results have been compared to the asymptotic solution shown in the publication by Garandet *et al.* [1991] and to the numerical solution obtained by Moreau *et al.* [1994] by use of a commercial code. Both show a quite interesting agreement.

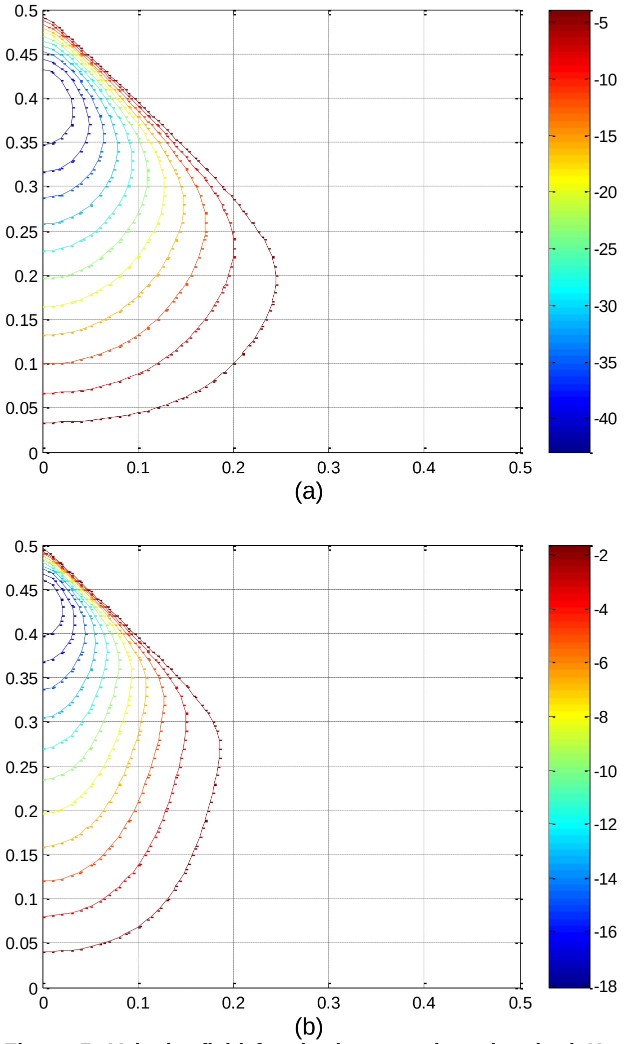


Figure 5: Velocity field for the lozenge-shaped rod: a) Ha =50 ; and (b) Ha = 100

6. Conclusion

In this work, a numerical finite element method for the solution of the magneto hydrodynamic problem associated with the study of the electromagnetic damping of the natural convection in the crystal growth horizontal Bridgman configuration is presented. The resulting code is first validated through comparisons of its numerical results with analytical ones obtained without any magnetic field. Numerical results have then been obtained for three cross-section shapes: circular, square, and lozenge. For a given Grashoff number, *Gr*, the Hartmann number, *Ha*, has been systematically changed to put into evidence the electromagnetic damping and the associated building of the Hartmann and the parallel layers. A few results are presented here. More of them will be presented at the congress.

References

[1] Carruthers, J.R., "Crystal growth from the melt", N.B. Hannay (Ed.), Treaties on Solid State Chemistry, vol. 5, Plenum, New York, 1975, pp.325-406.

[2] Hurle, D.T.J., "Hydrodynamics in crystal growth", in: E. Kaldis, H.J. Scheel (Eds.), Crystal Growth and Materials, North-Holland, Amsterdam, 1977, pp.550-569.

[3] Kobayashi, N., "Heat transfer in Czochralski crystal growth", in: W.R. Wilcox (Ed.), Preparation and Properties of Solid State Materials, vol. 6, Basel/Marcel Dekker, New York, 1981, pp. 119-253.

[4] Langlois, W.E, "Buoyancy-driven flows in crystal-growth melt", Annual Review Fluid Mechanics 1985,17:191-215.

[5] Garandet J. P., Alboussiére ,T., and Moreau, R., "Buoyancy driven convection in a rectangular enclosure with a transverse magnetic field", J. of Heat and Mass Transfer , 1991;35:4:741-748.

[6] Alchaar, S., Vasseur, P. and Bilgen, E., Hydromagnetic natural convection in a titled rectangular porous enclosure", Num. Heat Transfer, Part A, 1995;27:107-127.

[7] Ben Hadid, H., Henry, D., "Numerical simulation of convective three-dimensional flows in a horizontal cylinder under the action of a constant magnetic field", Journal of Crystal Growth , 1996;166:436-445.

[8] Davoust, L., Moreau, R., Cowley, M.D, Tanguy, P.A, and Bertrand, F., "Numerical and analytical modelling of MHD driven flow in a Bridgman crystal growth configuration", Journal of Crystal Growth, 1995:422-432.

[9] Bũhler, L., "Magneto-convection in long vertical channels", in: Proceedings of the Aussois Conference of MHD Flows, Aussois, France 1997, pp. 391-396,

[10] Moreau, R., Alboussière, T., Ben Salah, N., Garandet, J.P., Bolcato, R. and Bianchi, A.M., "MHD Control of Free Convection in Horizontal Bridgman Crystal Growth", Magnetohydrodynamics (Magnitnaya Gidrodinamika), 1994;30:3:282-288.

[11] http://fluxexpert.astek.fr

[12] .Bejan, A., Tien, C.L., "Fully developed natural counter flow in a long horizontal pipe with different end temperatures", Int. J. Heat Mass Transfer, 1977; 21:6:701-70.